Bilinear Hilbert $\longleftrightarrow$ An outer Carleson embedding :

4.1

Lacey and Thiele:

$$\left\| \int_{\mathbb{R}} f_1(x-t)\, f_2(x+t)\,\frac{dt}{t}\right\|_{L^{p_3'}(\mathbb{R})} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R})}\,\|f_2\|_{L^{p_2}(\mathbb{R})}$$

when $2<p_i<\infty$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3'}=1-\frac{1}{p_3}$

Trilinear form:

$$\Lambda(f_1,f_2,f_3) := p.v.\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\prod_{j=1}^{3} f_j(x-\beta_j t)\,dx\right)\frac{dt}{t}$$

Need:

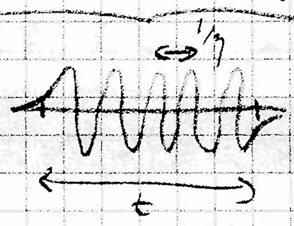
$$|\Lambda(f_1,f_2,f_3)| \lesssim \|f_1\|_{L^{p_1}}\,\|f_2\|_{L^{p_2}}\,\|f_3\|_{L^{p_3}}\,.$$

for $\beta=(1,-1,0)$.

- FBI type transform:

$$F(y,\eta,t) := \int_{\mathbb{R}} f(x)\,\underbrace{e^{i\eta(y-x)}\,\frac{1}{t}\,\varphi\!\left(\frac{y-x}{t}\right)}\,dx$$

Assumptions on φ:



$\hat\varphi \geq 0$

$\hat\varphi(0)\neq 0$

supp $\hat\varphi \subset [-\varepsilon,\varepsilon]$ small fixed

- Λ in terms of F_1, F_2, F_3:

Three orth. vectors in $\mathbb{R}^3$: $(1,1,1)$, $\underset{\sim}{\beta}$ and $\underset{\sim}{\alpha}$. $\qquad (-1,1,0) \quad (1,1,-2)$

$$\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\prod_j \underbrace{\int_{\mathbb{R}} f_j(x_j)\, e^{i(\alpha_j\eta+\beta_j t^{-1})(y-x_j)}\,\frac{1}{t}\,\varphi\!\left(\frac{y-x_j}{t}\right)dx_j}_{=F_j(y,\,\alpha_j\eta+\beta_j t^{-1},\,t)}\right)dy\,d\eta\,dt$$

$$/\; e^{iy(\alpha_1+\alpha_2+\alpha_3)\eta - i(\beta_1+\beta_2+\beta_3)t^{-1}} = 1 \;/$$

$$=\iint dy\,dt \int d\eta\, \underbrace{\int e^{-i\langle \eta\alpha, x\rangle}\,\underbrace{\prod_j f_j(x_j)\,e^{-i\beta_j t^{-1}x_j}\,\frac{1}{t}\,\varphi\!\left(\frac{y-x_j}{t}\right)}_{g(\bar{x})}\,d\bar{x}}_{\hat{g}(\eta\alpha)}$$

$$/ \int \hat{g}(\eta\alpha)\, d\eta = \iint g(u(1,1,1) + v\beta)\, du\, dv \,/$$

$$= \iint dy\, dt \iint du\, dv \prod_j f_j(u+\beta_j v)\, e^{-ivt^{-1}} \frac{1}{t}\, \varphi\left(\frac{y-u-\beta_j v}{t}\right)$$

$$/ \sum_j \beta_j (u+\beta_j v) = v \quad \text{if} \quad |\beta|_2 = 1 \,/$$

$$= / \int dy \prod_j \varphi\left(\frac{y-u-\beta_j v}{t}\right) = / \frac{y-u}{t} = z / = t \int dz \prod_j \varphi\left(z - \beta_j \frac{v}{t}\right) =: \Psi\left(\frac{v}{t}\right) /$$

$$= \iint \underbrace{\left(\int \prod_j f_j(u+\beta_j v)\, du\right)}_{=: g(v)} e^{-ivt^{-1}} \frac{1}{t^2} \Psi\left(\frac{v}{t}\right) dv\, dt$$

$$= \iint g(v)\, e^{-ivs}\, \Psi(vs)\, dv\, ds$$

Missing Dirac term

$$/ \begin{matrix} v' = v \\ s' = vs \end{matrix} \quad \frac{dv's'}{dvs} = \begin{vmatrix} 1 & 0 \\ s & v \end{vmatrix} = v = v' /$$

$$= \iint g(v)\, e^{-is}\, \Psi(s)\, \frac{1}{v}\, dv\, ds = \hat{\Psi}(1) \int g(v) \frac{dv}{v}$$

$$/ \hat{\Psi}(\eta) = \int dx \int dz \underbrace{\prod_j \varphi(z - \beta_j x)}_{= g(z(1,1,1)+\beta x)} e^{-i\eta x} \quad \to \beta_j \eta (z - \beta_j x)$$

$$g(\bar{x}) = \varphi(x_1)\varphi(x_2)\varphi(x_3)\, e^{-i\langle \eta\beta, \bar{x}\rangle}$$

$$= \int \hat{g}(w\alpha)\, dw = \int \prod \hat{\varphi}(w\alpha - \eta\beta)\, dw > 0 \,/$$

Thus, need:

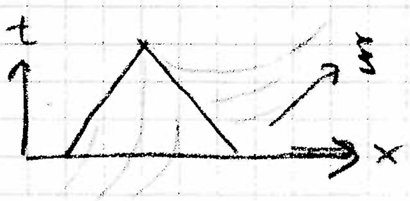
$$\left| \iiint \prod_j F_j(y, \underset{\text{mem}}{\underline{\underline{\alpha_j}}}\eta + \beta_j t^{-1}, t)\, d\eta\, dy\, dt \right| \lesssim \prod_j \|f_j\|_{L^{p_j}}$$

This is proved with outer Hölder & outer Carleson similar to seminar 3.

"Wavepackage tents" $\mathbb{E}$:

Heisenberg!

$$T(x,\xi,t) := \{(y,\eta,s);\ 0<s<t,\ |y-x|<t,\ |\eta-\xi|<\tfrac{1}{s}\}$$

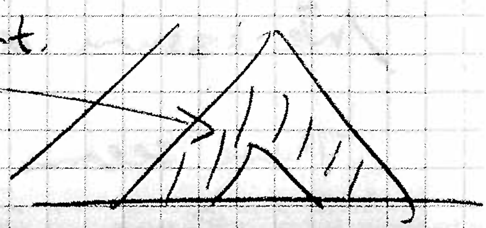


Premeasure: $\sigma(T(x,\xi,t)) := t$

Size:

$$S(f, T(x,\eta,t)) := \frac{1}{t}\iiint_{T(x,\eta,t)} |f| \, dy\, d\eta\, ds$$

Need sizes S_1, S_2, S_3 to use for Hölder.
Since $\int \varphi \neq 0$, an L_2-size Carleson estimate is expected to "non-solid" tent.



$$T^{(j)}(x,\xi,t) := \{ (y,\eta,s) ;\ 0<s<t,\ |y-x| < t-s,\ |(\eta-\xi) - \beta_j t^{-1}| \le b|\alpha_j| t^{-1} \}$$

to be chosen small.

$T^{(1)} \cap T^{(2)} = \emptyset$:

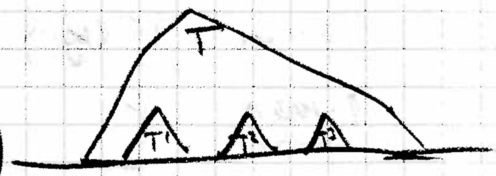
$$|\beta_1 - \beta_2| \le t\,|(\eta-\xi-\beta_1 t^{-1}) - (\eta - \xi - \beta_2 t^{-1})| \le b(|\alpha_1| + |\alpha_2|)$$

We assume $\beta_1, \beta_2, \beta_3$ all distinct.
$\Rightarrow$ $T^{(j)}$ pairwise disjoint if $b>0$ small.

$$S(f_1 f_2 f_3, T) = \frac{1}{t}\int_T |f_1 f_2 f_3| = \frac{1}{t}\int_{T\setminus(T^1\cup T^2\cup T^3)} + \sum_j \frac{1}{t}\int_{T\cap T^j}$$

$$\le \frac{1}{t}\prod_j \Big(\int_{T\setminus T_j} |f_j|^3\Big)^{1/3} + \frac{1}{t}\sum_j \sup_T |f_j| \prod_{k\neq j} \Big(\int_{\underbrace{T\setminus T_k}_{\text{by disjointness!}}} |f_k|^2\Big)^{1/2}$$

/ $L_2, L_\infty \to L_3$ interpolation /



$$\lesssim \prod_j \Bigg(\underbrace{\Big(\frac{1}{t}\int_{T\setminus T^j} |f_j|^2\Big)^{1/2} + \sup_T |f_j|}_{=:\ S_j(f_j, T)} \Bigg)$$

4:4

Outer Hölder $\Rightarrow$

$$|\Delta(f_1, f_2, f_3)| \lesssim \prod_j \| \overbrace{F_j(y, \alpha_j \eta + \beta_j s^{-1}, s)}^{= F_j \circ \Phi_j(y,\eta,s) =: G_j} \|_{L^{p_j}(\mathbb{R}^3_+, \sigma, S_j)}$$

Change of variables:

$$\Phi_j : \mathbb{R}^3_+ \to \mathbb{R}^3_+ : (y, \eta, s) \longmapsto (y, \alpha_j \eta + \beta_j s^{-1}, s) = (x, \xi, t)$$

$$\Phi_j^{-1} : (x, \xi, t) \longmapsto (x, \alpha_j^{-1}\xi - \alpha_j^{-1}\beta_j t^{-1}, t)$$

/We assume $\alpha_j \neq 0$./

There seems to be some error on p. 295 ...

Use Prop. 3.2: a tent is mapped to a similar tent with same height.

It remains to prove the main Carleson estimate:

__Thm 5.1__:

$$\| F(x, \xi, t) \|_{L^p(\mathbb{R}^3_+, \sigma, S^b)} \lesssim \| f \|_{L_p(\mathbb{R}, dx)}$$

for $2 < p \leq \infty$.

$$S^b(F, T_{\alpha\beta}) := \left(\frac{1}{s} \int_{T_{\alpha\beta} \setminus T^b} |F|^2 \, dy \, d\eta \, ds \right)^{1/2} + \sup_{T_{\alpha\beta}} |F|$$

$T_{\alpha\beta}$: $|\alpha(\eta - \xi) + \beta t^{-1}| \leq \frac{1}{t}$

T^b : $|\eta - \xi| \leq \underbrace{b}_{\text{small}} \frac{1}{t}$

Proof by interpolation weak $L_2 - L_\infty$.

__$p = \infty$__: Need estimate $S^b(F, T(x, \xi, t)) \lesssim \| f \|_\infty$

Here $|F(y, \eta, s)| \leq \|\varphi\|_1 \|f\|_\infty$, so

__Need__: $$\int_{T_{\alpha\beta} \setminus T_b} |F|^2 \lesssim \cancel{\|f\|_\infty^2} \text{ first } \|f\|_2^2$$

$$\text{LHS} \lesssim \int_0^\gamma ds \int_{\mathbb{R}} dy \int_{\frac{b}{s} \le |\eta - \xi| \le \frac{c}{s}} |F|^2 \lesssim / \eta - \xi + \gamma \tfrac{1}{s} / \qquad \text{(4:5)}$$

$$\lesssim \int_0^\gamma \frac{ds}{s} \int_{\mathbb{R}} dy \int_b^c d\gamma \, |\underbrace{F(y, \xi + \gamma \tfrac{1}{s}, s)}_{= \int f(x) \underbrace{e^{i(\xi + \gamma \frac{1}{s})(y-x)} \frac{1}{s} \varphi\left(\frac{y-x}{s}\right)}_{=: \overline{\varphi_{y,\gamma,s}(x)}} dx}|^2$$

$\uparrow$ Symmetry

$\operatorname{supp} \hat{\varphi} \subset [-\varepsilon, \varepsilon]$

$\operatorname{supp} \frac{1}{s} \varphi\left(\frac{y-\cdot}{s}\right) \subset \left[-\frac{\varepsilon}{s}, \frac{\varepsilon}{s}\right]$

$\operatorname{supp} \varphi_{y,\gamma,s} e^{-i\xi x} \subset \left[-\frac{\varepsilon}{s} + \frac{\gamma}{s}, \frac{\varepsilon}{s} + \frac{\gamma}{s}\right] \not\ni 0$ if

$\varepsilon < b \le \gamma$

Fix $\gamma \ge b$:

$$X^2 = \Big(\underbrace{\int_0^\infty \int_{\mathbb{R}} |\langle f, \varphi_{y,\gamma,s} \rangle|^2 \, dy \frac{ds}{s}}_{= \langle \langle f, \varphi_{y,\gamma,s} \rangle \varphi_{y,\gamma,s}, f \rangle} \Big)^2$$

C-S

Assume $\|f\|_2 = 1$.

$$\le \left\| \int_0^\infty \int_{\mathbb{R}} \langle f, \varphi_{y,\gamma,s} \rangle \varphi_{y,\gamma,s} \, dy \frac{ds}{s} \right\|^2 \cdot \cancel{\|f\|_2^2}$$

$$\int_0^\infty \frac{ds}{s} \int_{\mathbb{R}} dy \int_0^\infty \frac{dr}{r} \int_{\mathbb{R}} dz \, |\langle f, \varphi_{y,\gamma,s} \rangle \langle \varphi_{y,\gamma,s}, \varphi_{z,\gamma,r} \rangle \langle \varphi_{z,\gamma,r}, f \rangle|^2$$

~~May assume $|\langle f, \varphi_{z,\gamma,r} \rangle| \le |\langle f, \varphi_{y,\gamma,s} \rangle|$~~

$$\lesssim \int_0^\infty \frac{ds}{s} \int_{\mathbb{R}} dy \, |\langle f, \varphi_{y,\gamma,s} \rangle|^2 \underbrace{\left(\int_0^\infty \frac{dr}{r} \int_{\mathbb{R}} dz \, |\langle \varphi_{y,\gamma,s} e^{-i\xi(\cdot)}, \varphi_{z,\gamma,r} e^{-i\xi(\cdot)} \rangle| \right)}_{\underbrace{\int_{r \ge t} + \int_{r \le t}}_{\text{Need } \lesssim 1}}$$

$(\lesssim X^2)$

<u>$r \ge t$ case</u>:

$\varphi_{y,\gamma,s} e^{-i\xi(\cdot)}$: Schwartz, $\int (\cdot) = 0$, scale s,

L^∞ norm $\frac{1}{s}$, position y

primitive: scale s, L^∞ norm 1, pos. y

derivative: scale s, L^∞ norm $\frac{1}{s^2}$, pos. y.

IBP $\Rightarrow$

$$\int_{\mathbb{R}} dz \int_s^\infty \frac{dr}{r} \int_{\mathbb{R}} dx \, \frac{1}{\left(1+\frac{|y-x|}{s}\right)^2} \, \frac{1/r^2}{\left(1+\frac{|z-x|}{r}\right)^2}$$

$$\lesssim \int_s^\infty \frac{dr}{r^2} \underbrace{\int_{\mathbb{R}} dx \, \frac{1}{\left(1+\frac{|y-x|}{s}\right)^2}}_{\lesssim s} \lesssim 1$$

<u>$r \le t$ case</u>: opposite IBP $\Rightarrow$

$$\int_{\mathbb{R}} dz \int_0^s \frac{dr}{r} \int_{\mathbb{R}} dx \, \frac{1/s^2}{\left(1+\frac{|y-x|}{s}\right)^2} \, \frac{1}{\left(1+\frac{|z-x|}{r}\right)^2}$$

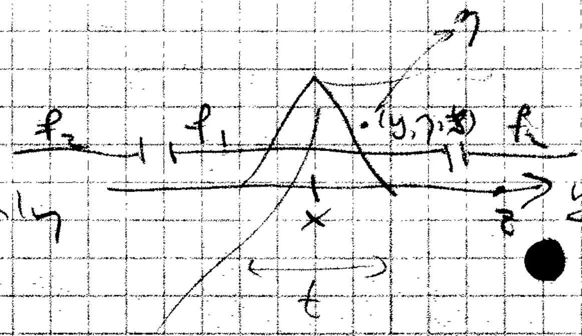
$$\lesssim \int_0^s \frac{dr}{r} \frac{1}{s^2} s \cdot r \lesssim 1.$$

Have shown: $\int_{T_{ap}\setminus T^b} |F|^2 \lesssim \|f\|_2^2$

Now localise to an L^∞ estimate:

$f = f_1 + f_2$

$f_1 = f \cdot 1_{[x-2t,\, x+2t]}$



$\|f_1\|_2^2 \le t \cdot \|f\|_\infty^2$, so need only

$$\int_{T_{ap}\setminus T^b} |F_2|^2 \lesssim t\|f\|_\infty^2$$

$$|F_2(y,\eta,s)| \le \int_{\cancel{\mathbb{R}}\ [-t,t]^c} |f_2(y-z)| \, \frac{1}{s} \left|\varphi\left(\frac{z}{s}\right)\right| dz$$

$$\lesssim \|f\|_\infty \underbrace{\int_{t/s}^\infty |\varphi(z)| \, dz}_{\lesssim \frac{s}{t}}$$

$$\Rightarrow \int_{T_{ap}\setminus T^b(x,\xi,t)} |F_2|^2 \lesssim \int_0^t ds \int_{|y-x|<t-s} dy \underbrace{\int_{|\eta-\xi|<\frac{1}{s}} d\eta}_{\lesssim \frac{1}{s}} \left(\frac{s}{t}\|f\|_\infty\right)^2$$

$$\lesssim \int_0^t ds \underbrace{(t-s)}_{\le t} \frac{s}{t^2} \|f\|_\infty^2 \lesssim t\|f\|_\infty^2$$

$p = 2$

For technical reasons, we work with subcollection $\mathbb{E}_\Delta \subset \mathbb{E}$ consisting of tents $T(x,\xi,t)$ with

$$(x,\xi,t) \in X_\Delta := \{ (2^{k-4} n, 2^{-k-8} b \ell, 2^k) \in \mathbb{R}^3_+ ; \; n, \ell, k \in \mathbb{Z} \}$$

fixed small as before.

$= T_{\alpha\beta}$ α, β fixed

We have

$\mu_\Delta \approx \mu$

$\| \cdot \|_{L^p(\mathbb{R}^3_+, \sigma, S^b)} \approx \| \cdot \|_{L^p(\mathbb{R}^3_+, \underbrace{\sigma_\Delta, S^b_\Delta}_{\text{restricts to } E_\Delta})}$

May and will assume $\operatorname{supp} \hat{f}$ compact & $\|f\|_2 = 1$

Need: $\forall \lambda > 0$: construct $Q \subset X_\Delta$ s.t.

$$\sum_{(x,\xi,t) \in Q} t \lesssim \lambda^{-2}$$

where $S^b(F 1_{\mathbb{R}^3_+ \setminus E}, T') \leq \lambda$, $\forall T' \in E_\Delta$

$$E := \bigcup_{(x,\xi,t) \in Q} T(x,\xi,t)$$

$$\left(\Rightarrow \mu(S(F) > \lambda) \leq \mu(E) \lesssim \lambda^{-2} \underbrace{\|f\|_2^2}_{=1} \right)$$

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① Aim first to construct $Q_0 \subset X_\Delta$ s.t.

$$E := \bigcup_{T \in Q_0} T \Rightarrow |F(y,\eta,s)| \leq \lambda \text{ on } \mathbb{R}^3_+ \setminus E.$$

C-S $\Rightarrow |F(y,\eta,s)| \leq \frac{1}{\sqrt{s}} \|f\|_2 \|\varphi\|_2 \lesssim \frac{1}{\sqrt{s}}$

Lemma 5.2 $\forall (y,\eta,s) \in \mathbb{R}^3_+$ $\exists (x,\xi,t) \in X_\Delta$

s.t. $2^{-3} t < s \leq 2^{-2} t$

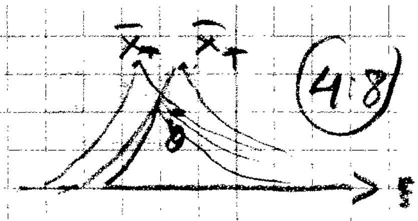
$|y - x| \leq 2^{-4} t$

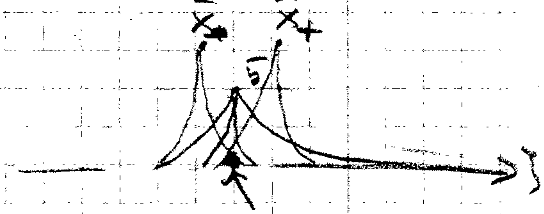
$|\eta - \xi| \leq 2^{-8} b \frac{1}{t}$

$\exists\ \xi_\pm : (x, \xi_\pm, t) \in X_\Delta$ (4.8)

✓ $T(y,\eta,s) \subset T(x,\xi_+,t) \cup T(x,\xi_-,t)$

$\underbrace{T(y,\eta,s)}_{b} \cap T^b\underbrace{(x,\xi_+,t)}_{\bar{x}_+} \cap T^b\underbrace{(x,\xi_-,t)}_{\bar{x}_+} \subset T^b(y,\eta,s)$





Pf $t := 2^k \in [2^2 s, 2^3 s)$

$x := 2^{k-4} \lfloor y / 2^{k-4} \rfloor$

$\xi_- = \xi := 2^{-k-8} b \lfloor \eta / 2^{-k-8} b \rfloor$

$\xi_+ := 2^{-k-8} b \lceil \eta / 2^{-k-8} b \rceil$

$(y',\eta',s') \in T(y,\eta,s) \Rightarrow$

$s' \leq s \leq 2^{-2} t$

$|y'-x| \leq |y'-y| + |y-x| \leq (s-s') + 2^{-4}t \leq t-s'$

Need $|\alpha(\eta'-\xi_\pm) + \beta \frac{1}{s'}| \leq \frac{1}{s'}$

Have $|\alpha(\eta'-\xi) + \beta \frac{1}{s'}| \leq \frac{1}{s'}$

Need $|\alpha \underbrace{(\xi_+ - \xi_-)}_{\leq 2^{-4-k} b = 2^{-4} b \frac{1}{t} \leq 2^{-6} b \frac{1}{s'}}| \leq \frac{1}{s'}$ ok

- $(y',\eta',s') \in T^b(x,\xi_+,t) \cap T^b(x,\xi_-,t)$

Need: $|\eta'-\eta| \leq b \frac{1}{s}$

Have: $|\eta'-\xi_\pm| \leq b \frac{1}{t} \leq 2^{-2} b \frac{1}{s}$

$|\eta-\xi_\pm| \leq 2^{-8} b \frac{1}{t} \leq 2^{-10} b \frac{1}{s}$

Write $(y,\eta,s) \mathring{\in} T(x,\xi,t)$ for the situation in the lemma

↳ Assume $(y_i,\eta_i,s_i) \mathring{\in} T(x_i,\xi_i,t_i)$, $i=1,\dots,n-1$ chosen

Choose $(y_n,\eta_n,s_n) \mathring{\in} T(x_n,\xi_n,t_n)$ s.t.

$(y_n,\eta_n,s_n) \in \mathbb{R}^3_+ \setminus (T_1 \cup \dots \cup T_{n-1})$

$|F(y_n,\eta_n,s_n)| > \lambda$

t_n is maximal

Claim: $\sum_1^\infty t_h \lesssim \lambda^{-2}$ (4:9)

$K_m := \{h;\ 2^m\lambda \le |F(y_h,\eta_h,s_h)| \le 2^{m+1}\lambda\}$

Need: $A := \sum_{h\in K_m} s_h |F(y_h,\eta_h,s_h)|^2 \le C$

$\left(\Rightarrow \sum \underset{\approx s_h}{t_h} \le \sum_m \sum_{h\in K_m} s_h \left(\frac{|F(y_h,\eta_h,s_h)|}{2^m\lambda}\right)^2 \lesssim \sum_m 2^{-2m}\lambda^{-2} \right)$

$\varphi_h(x) := e^{-i\eta_h(y_h-x)} \frac{1}{\sqrt{s_h}} \varphi\left(\frac{y_h-x}{s_h}\right)$

$\Rightarrow \sqrt{s_h}\, F(y_h,\eta_h,s_h) = \langle f, \varphi_h\rangle$

$A^2 = \Big(\underbrace{\sum_{h\in K_m} |\langle f,\varphi_h\rangle|^2}\Big)^2 \le \Big\| \sum_{h\in K_m} \langle f,\varphi_h\rangle \varphi_h \Big\|_2^2$

$= \langle \sum_{h\in K_m} \langle f,\varphi_h\rangle\varphi_h, f\rangle$

$= \sum_{h\,l} \langle f,\varphi_h\rangle\langle\varphi_h,\varphi_l\rangle\langle\varphi_l, f\rangle$

$\le 2 \sum_{t_l\le t_h} \left| \frac{\langle f,\varphi_h\rangle}{\sqrt{s_h}} \langle\varphi_h,\varphi_l\rangle \frac{\langle \varphi_l, f\rangle}{\sqrt{s_l}} \right| \sqrt{s_h s_l}$ $\quad \approx \left|\frac{\langle f,\varphi_h\rangle}{\sqrt{s_h}}\right|$

$\lesssim \sum_{h\in K_m} \Bigg(\underbrace{\sum_{\substack{l\in K_m\\ t_l\le t_h}} \sqrt{\frac{s_l}{s_h}} |\langle\varphi_h,\varphi_l\rangle|}_{\text{Claim } \lesssim 1} \Bigg) |\langle f,\varphi_h\rangle|^2 \lesssim A$

$\Rightarrow A$ bounded

Assume $t_l \le t_h$, $\langle\varphi_h,\varphi_l\rangle \ne 0$

$\Rightarrow \operatorname{supp}\hat\varphi_h \cap \operatorname{supp}\hat\varphi_l \ne \emptyset$

$\Rightarrow \eta_h + \delta b s_h^{-1} = \eta_l + \gamma b s_l^{-1}$ for small δ, γ.

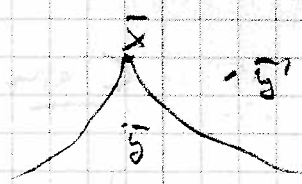
/Assume $\operatorname{supp}\hat\varphi \subset [2^{-\delta}b, 2^{\delta}b]$ /

Similarly assume $t_{l'} \le t_h$, $\langle\varphi_h,\varphi_{l'}\rangle \ne 0$

$\Rightarrow \eta_h + \delta' b s_h^{-1} = \eta_{l'} + \gamma' b s_{l'}^{-1}$

Wlog T_l selected before $T_{l'}$: $t_{l'} \le t_l$

$(y_\ell, \eta_\ell, s_\ell) \overset{\circ}{\in} T_\ell \not\ni (y_{\ell'}, \eta_{\ell'}, s_{\ell'})$



4:10

$$\Rightarrow \ |\alpha(\eta_{\ell'} - \xi_\ell) + \beta s_{\ell'}^{-1}|$$
$$\le |\alpha(\eta_\ell - \xi_\ell)| + |\alpha(\eta_{\ell'} - \eta_\ell)| + |\beta s_{\ell'}^{-1}|$$
$$\le b t_\ell^{-1} + \underbrace{|\eta_\ell - \eta_{\ell'}|}_{\le 4 \cdot 2^{-8} \cdot s_{\ell'}^{-1}} + |\beta s_{\ell'}^{-1}| \le \xi_{\ell'}^{-1}$$

Claim: $[x_\ell - 2^{-8} t_\ell, x_\ell + 2^{-8} t_\ell] \cap [x_{\ell'} - 2^{-8} t_{\ell'}, x_{\ell'} + 2^{-8} t_{\ell'}] = \emptyset$

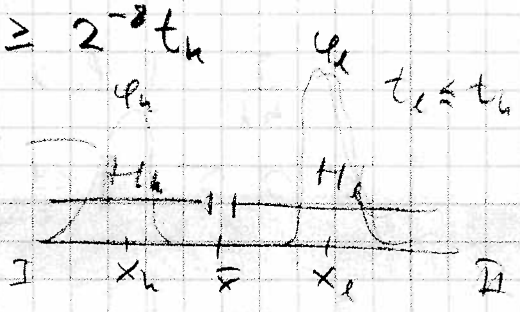
$\hookrightarrow$ If not: $|y_{\ell'} - x_\ell| \le |y_{\ell'} - x_{\ell'}| + |x_{\ell'} - x_\ell|$

$$\le 2^{-4} t_{\ell'} + \underset{\substack{\uparrow \\ \text{(by claim)}}}{2^{-4} t_\ell} \quad < t_\ell - s_{\ell'} \quad (t_{\ell'} < t_\ell)$$

$\Rightarrow (y_{\ell'}, \eta_{\ell'}, s_{\ell'}) \in T_\ell$ contradiction.

$\ell' = k \Rightarrow |x_k - x_\ell| \ge 2^{-8}(t_k + t_\ell) \ge 2^{-8} t_k$

The needed Schur estimate: for



$$|\langle \varphi_k, \varphi_\ell \rangle|$$
$$\lesssim \|\varphi_k\|_{L^1(H_k)} \|\varphi_\ell\|_{L^\infty(H_k)} + \|\varphi_k\|_{L^\infty(H_\ell)} \|\varphi_\ell\|_{L^1(H_\ell)}$$
$$\lesssim \sqrt{t_k}\left(\frac{1}{\sqrt{t_\ell}} \varphi\left(\frac{x_k - x_\ell}{t_\ell}\right)\right) + \left(\frac{1}{\sqrt{t_k}} \varphi\left(\frac{x_k - x_\ell}{t_k}\right)\right)\sqrt{t_\ell}$$
$$\underset{\substack{\uparrow \\ \text{claim}}}{\lesssim} \frac{1}{\sqrt{t_k t_\ell}} \underbrace{\int_{x_\ell - 2^{-8} t_\ell}^{x_\ell + 2^{-8} t_\ell} \frac{dx}{1 + \left(\frac{x - x_k}{t_k}\right)^2}}_{\gtrsim \ t_\ell \frac{1}{1 + \left(\frac{x_k - x_\ell}{t_k}\right)^2}} \qquad / |x_k - x_\ell| \ge t_k \ge t_\ell /$$

$\hookrightarrow$ • $\sqrt{\frac{t_k}{t_\ell}} \varphi\left(\frac{x_k - x_\ell}{t_\ell}\right) \le \sqrt{\frac{t_\ell}{t_k}} \frac{1}{\cancel{1+}\left(\frac{x_k - x_\ell}{t_k}\right)^2}$ ≥ 1

$$\Leftrightarrow \varphi\left(\frac{x_k - x_\ell}{t_\ell}\right) \le \frac{t_\ell^2}{(x_k - x_\ell)^2} \frac{t_k}{t_\ell} \quad \text{ok}$$

• $\sqrt{\frac{t_\ell}{t_k}} \varphi\left(\frac{x_k - x_\ell}{t_k}\right) \le \sqrt{\frac{t_\ell}{t_k}} \frac{1}{1 + \left(\frac{x_k - x_\ell}{t_k}\right)^2}$ ok

(2) Similar selection of tents with large L_2-size.

$\xi \in \mathbb{R}$ given.

$X = \underset{\eta \geq \xi}{X_\xi^+} \cup \underset{\eta < \xi}{X_\xi^-}$

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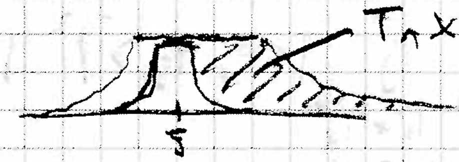
Right half-tents:



$(x, \xi, t) \in X_\Delta$ bad if

$$\frac{1}{t} \iint\limits_{\substack{T(x,\xi,t) \cap X_\xi^+ \cap \\ (T^b(x,\xi,t))^c \cap E^c}} |F|^2 \, dy \, d\eta \, ds \geq 2^{-8} \lambda^2$$

The L_2-est. $\Rightarrow$ upper bound for t of bad points.

$\Rightarrow \xi$ = multiple of $2^{-8-\ldots} b$.

Selection:

- (x_1, ξ_1, t_1) : ξ_1 maximal, and among these: t_1 maximal

  $\to T_1$ & T_1^b , $X_1^+ = X_{\xi_1}^+$

- $E_2 := E \cup T_1$

  Find $(x_2, \xi_2, t_2) \in X_\Delta$ with

  $$\frac{1}{t_2} \int\limits_{\substack{T(x_2,\xi_2,t_2) \cap X_\xi^+ \\ \cap (T^b(x_2,\xi_2,t_2))^c \cap E_2^c}} |F|^2 \, dy \, d\eta \, d\xi \geq 2^{-8} \lambda^2$$

  and ξ_2 maximal, and among these t_2 maximal.

  $\leadsto T_2, T_2^b$, $X_2^+ = X_{\xi_2}^+$

- ---

$T_n^* := T_n \cap X_n^+ \setminus (T_n^b \cup E_n)$

Claim: $\sum_1^\infty t_n \leq C \lambda^{-2}$

Need: $\sum_1 \underbrace{\int\limits_{T_n^*} |F|^2 \, dy \, d\eta \, ds}_{\lambda^2 t_n \leq} \leq C$

C-S estimate:

$$A^2 := \Big(\sum_k \int_{T_k^*} \underbrace{|\langle f, \varphi_{\tilde{\zeta}} \rangle|^2}_{\langle f, \varphi_{\tilde{\zeta}} \rangle \langle \varphi_{\tilde{\zeta}}, f \rangle} \overbrace{dy\, d\eta \frac{ds}{s}}^{=: d\tilde{\zeta}} \Big)^2$$

$$\le \sum_{k,l} \int_{\substack{T_k^* \times T_l^* \\ s \le B s' \\ s' \le B s}} |\langle f, \varphi_{\tilde{\zeta}} \rangle|^2 \, |\langle \varphi_{\tilde{\zeta}}, \varphi_{\tilde{\zeta}'} \rangle| \, d\tilde{\zeta}\, d\tilde{\zeta}'$$

$$+ \sum_{k,l} \int_{\substack{T_k^* \times T_l^* \\ B s' \le s}} |\langle f, \varphi_{\tilde{\zeta}} \rangle| \, |\langle f, \varphi_{\tilde{\zeta}'} \rangle| \, |\langle \varphi_{\tilde{\zeta}}, \varphi_{\tilde{\zeta}'} \rangle| \, d\tilde{\zeta}\, d\tilde{\zeta}'$$

$$=: \mathrm{I} + \mathrm{II}$$

I: Need Schur:

$$\sup_{k, \tilde{\zeta} \in T_k^*} \sum_l \int_{\substack{\tilde{\zeta}' \in T_l^* \\ t' \sim t}} |\overbrace{\langle \varphi_{\tilde{\zeta}}, \varphi_{\tilde{\zeta}'} \rangle}^{= \langle \hat{\varphi}_{\tilde{\zeta}}, \hat{\varphi}_{\tilde{\zeta}'} \rangle}| \, d\tilde{\zeta}'$$

First $\int_{t/B}^{tB} \frac{dt'}{t'} \lesssim c$

Then $\int dz' \underbrace{\int d\eta' |\langle \hat{\varphi}_{\tilde{\zeta}}, \hat{\varphi}_{\tilde{\zeta}'} \rangle|}_{= 0 \text{ if } |\eta' - \eta| \gtrsim \frac{1}{t}}$

$$\lesssim \frac{1}{t} \sup_{\eta'} \underbrace{\int dy' |\langle \varphi_{\tilde{\zeta}}, \varphi_{\tilde{\zeta}'} \rangle|}_{\lesssim t \cdot 1 \cdot 1} \lesssim 1$$

II: $$= \underbrace{\sum_k \int_{T_k^*} d\tilde{\zeta} \, |\langle f, \varphi_{\tilde{\zeta}} \rangle| \Big(\sum_l \int_{\substack{T_l^* \\ B s' \le s}} |\langle f, \varphi_{\tilde{\zeta}'} \rangle \langle \varphi_{\tilde{\zeta}}, \varphi_{\tilde{\zeta}'} \rangle| \, d\tilde{\zeta}' \Big)}$$

$$\le \underbrace{\Big(\int_{T_k^*} |\langle f, \varphi_{\tilde{\zeta}} \rangle|^2 d\tilde{\zeta} \Big)^{1/2}}_{\le t_k \lambda^2} \underbrace{\Big(\int_{T_k^*} \Big(\sum_l \int_{\substack{T_l^* \\ B s' \le s}} |\langle f, \varphi_{\tilde{\zeta}'} \rangle \langle \varphi_{\tilde{\zeta}}, \varphi_{\tilde{\zeta}'} \rangle| \, d\tilde{\zeta}' \Big)^2 d\tilde{\zeta} \Big)^{1/2}}_{=: H_k}$$

$|F| \le \lambda$ on $T_k^* \subset E^c$ by ① $\Rightarrow$ (4.13)

$\overset{\shortparallel}{|\langle f, \varphi_{\bar\zeta'}\rangle| \cdot \frac{1}{\sqrt{s'}}}$

$$H_k \le \int_{T_k^*} d\bar\zeta \Big(\sum_{\substack{l \\ Bs'<s}} \int_{T_l^*} \sqrt{s'}\,\lambda\, \underbrace{|\langle \varphi_{\bar\zeta}, \varphi_{\bar\zeta'}\rangle|}_{\text{assume } \neq 0}\, d\bar\zeta' \Big)^2$$

$$\Rightarrow \eta' + \gamma' \frac{1}{s'} = \eta + \gamma \frac{1}{s} \quad \text{for } \gamma, \gamma' \in [-2^{-8}b, 2^{-8}b)$$

$$s' \le \frac{1}{B} s \le s \Rightarrow |\eta' - \eta| \le 2^{-4} b \frac{1}{s'}$$

$$\bar\zeta \in T_k^* \subset T_k \Rightarrow \alpha(\eta - \xi_k) + \beta \frac{1}{s} \le \frac{1}{s}$$

$$\bar\zeta' \in T_l^* \subset T_l \setminus T_l^b \cap E_h \Rightarrow \eta' - \xi_l \ge b \frac{1}{s'}$$

$\frac{1}{\xi_l}$

$$\Rightarrow \xi_k - \xi_l = (\eta - \eta') - (\eta - \xi_k) + (\eta' - \xi_l)$$

$$\ge -2^{-4} b \frac{1}{s'} - \frac{1}{\alpha}(1-\beta)\frac{1}{s} + b\frac{1}{s'}$$

$$= (1-2^{-4}) b \frac{1}{s'} - \frac{1}{\alpha}(1-\beta) \underbrace{\frac{1}{s}}_{\le \frac{1}{B}\frac{1}{s'}} \quad \ge 0 \quad \text{if } B \text{ large}$$

T_k chosen before T_l.

$\bar\zeta' \in T_l^* \subset (T_k)^c$ &

$$|\alpha(\eta' - \xi_k) + \beta\frac{1}{s'}| \le |\alpha(\eta' - \eta)| + |\alpha(\eta - \xi_k) + \beta \frac{1}{s}|$$

$$+ |\beta \frac{1}{s'} - \beta \frac{1}{s}| \le 2^{-4} b \frac{1}{s'} + \underbrace{\frac{1}{s}}_{\ll \frac{1}{s'}} + \underbrace{|\beta \frac{1}{s}|}_{\ll \frac{1}{s'}} + |\underbrace{\beta}_{<0.9} \frac{1}{s'}|$$

$$\le \frac{1}{s'}$$

$$|y' - x_k| \ge t_k - s' \ge t_k - s$$

Assume now $\bar\zeta'' \in T_{l'}^*$ with $t'' \le \frac{1}{B} t$ &

$\langle \varphi_{\bar\zeta}, \varphi_{\bar\zeta''} \rangle \neq 0$

Assume $t'' \le \frac{1}{B} t'$

$$\Rightarrow |\eta'' - \eta| \le 2^{-4} b \frac{1}{s''}$$

$$\Rightarrow |\eta'' - \eta'| \le 2^{-2} b \frac{1}{s''}$$

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$$\xi_\ell - \xi_{\ell'} = (\xi_\ell - \eta') + (\eta' - \eta'') + (\eta'' - \xi_{\ell'})$$
$$\geq - \underbrace{\tfrac{1-\beta}{\alpha}\tfrac{1}{s'}}_{\substack{\uparrow\\ \mathfrak{z}'\in T_\ell}} - \underbrace{2^{-2}b\tfrac{1}{s''}}_{\ll \frac{1}{s''}} + \underbrace{b\tfrac{1}{s''}}_{\mathfrak{z}''\in (T_{\ell'}^b)^c \cap E_{\ell'}} > 0$$

$\because$ T_ℓ chosen before $T_{\ell'}$.

$\mathfrak{z}'' \in (T_\ell)^c$

But

$$|\alpha(\eta''-\xi_\ell)+\beta\tfrac{1}{s''}| \leq |\alpha(\eta''-\eta')|$$
$$+|\alpha(\eta'-\xi_\ell)+\beta\tfrac{1}{s'}| + |\beta\tfrac{1}{s''}-\beta\tfrac{1}{s'}|$$
$$\leq 2^{-2}b\tfrac{1}{s''} + \tfrac{1}{s'} + \underbrace{|\beta\tfrac{1}{s''}|}_{\ll\frac{1}{s''}} + \underbrace{|\beta\tfrac{1}{s'}|}_{\ll\frac{1}{s''}} \leq \tfrac{1}{t''}$$

$$\Rightarrow |y''-x_\ell| > t_\ell - s'' > t_\ell - s' \geq |y'-x_\ell|$$

$$\Rightarrow \mathfrak{z}'' \neq \mathfrak{z}'.$$

Conclusion: Fix $\mathfrak{z}\in T_h^*$. For fixed $\mathfrak{z}'$

$$M = \{s' ;\ s' < \tfrac{1}{B}s,\ \exists \ell, \eta' \ (y',\eta',s')\in T_\ell^* \ \& \ \langle\varphi_{\mathfrak{z}},\varphi_{\mathfrak{z}'}\rangle \neq 0\}$$

Then $\frac{\sup M}{\inf M} \leq B$.

/ meas. smallness of above /

Call this t'-interval $I(y') = [T(y'), BT(y')]$.

$$\Rightarrow H_h \lesssim \int_{T_h^*} d\mathfrak{z} \Big(\int_{\substack{|y'-x_h|>t_h-s \\ \cancel{Bs'\leq s} \to I(y')}} dy' \int \frac{ds'}{s'} \underbrace{\sum_\ell \int_{\mathfrak{z}'\in T_\ell^*} d\eta'}_{=\int_{\mathfrak{z}'\in \cup_\ell T_\ell} \to \mathbb{R}} \lambda\sqrt{s'}\,|\langle\varphi_{\mathfrak{z}},\varphi_{\mathfrak{z}'}\rangle| \Big)^2$$

$$\leq \int_{T_h^*} d\mathfrak{z} \Big(\int_{|y'-x_h|>t_h-s} dy' \sup_{s'\in I(y')} \int_{\mathbb{R}} d\eta' \, \lambda\sqrt{s'}\, |\underbrace{\langle\varphi_{\mathfrak{z}},\varphi_{\mathfrak{z}'}\rangle}_{=\langle\hat\varphi_{\mathfrak{z}},\hat\varphi_{\mathfrak{z}'}\rangle\neq 0}| \Big)^2$$

only in $\frac{1}{s'}$ interval in η'

$$\lesssim \int_{T_k^*} d\xi \Big(\int_{|y'-x_k|>t_k-s} dy' \ \cancel{\sup_{s'\in I(y')}} \ \cancel{\sup_{\xi'}} \frac{1}{\sqrt{s'}} \underbrace{|\langle \varphi_{\xi}, \varphi_{\xi'}\rangle|}_{\lesssim \frac{1}{\sqrt{s}} \frac{1}{\left(1+\frac{|y'-y|}{s}\right)^2} \cdot \underbrace{\int|\varphi_{\xi'}|}_{\le \sqrt{s'}}} \Big)^2$$

$$\lesssim \int_{T_k^*} d\xi \Big(\underbrace{\int_{|y'-x_k|>t_k-s} \frac{\lambda}{\sqrt{s}} \frac{dy'}{\left(1+\frac{|y'-y|}{s}\right)^2}}_{} \Big)^2$$

$\boxed{4:15}$

$$= \frac{\lambda^2}{s}\Big(\int_{|sy''+y-x_k|>t_k-s} s \frac{dy''}{(1+|y''|)^2} \Big)^2$$

$$\Rightarrow s|y''| \ge t_k - s - |y-x_k|$$

$$\int_{|x-y|>e-1} \frac{dx}{(1+|x|)^2} \lesssim \min\Big(1, \frac{1}{e-|x|}\Big) \approx \frac{1}{1+e-|x|} \quad \text{if } |x|<e$$

$$\lesssim \int_{T_k^*} d\xi \, \lambda^2 s \Big(\frac{1}{1+\frac{t_k-|y-x_k|}{s}} \Big)^2$$

$$\lesssim \int_0^{t_k} \frac{ds}{s} \int_{x_k-t_k}^{x_k+t_k} dy \underbrace{\int_{\xi_k-2\alpha^{-1}\frac{1}{s}}^{\xi_k+2\alpha^{-1}\frac{1}{s}} d\xi \, \lambda^2 s}_{\lesssim \lambda^2} \frac{1}{\left(1+\frac{t_k-|y-x_k|}{s}\right)^2}$$

$$\lesssim \lambda^2 \int_0^{t_k} \frac{ds}{s} \int_0^{t_k} dy \frac{1}{\left(1+\frac{y}{s}\right)^2}$$

$$\lesssim \lambda^2 \int_0^{t_k} \frac{ds}{s} \underbrace{\int_0^{t_k/s} \frac{s\,dy''}{(1+y'')^2}}_{\le C\cdot s} \lesssim \lambda^2 t_k$$

Collect ests:

$$A^2 \le I + II \le C\cdot A + \sum_k \underbrace{\sqrt{t_k\lambda^2}\cdot\sqrt{\lambda^2 t_k}}_{= t_k\lambda^2 \lesssim \int_{T_k^*} |F|^2 d\xi} \lesssim A$$

$$\Rightarrow A \lesssim 1.$$

By possibly continuing this recursion for ordinals beyond ∞, we obtain a collection of tents Q_+ with

(4:16)

$$\sum_{T \in Q_+} \sigma(T) \lesssim \lambda^{-2}$$

s.t. for $E_+ := E \cup \bigcup_{T \in Q_+} T$ we have

$$\frac{1}{s} \int_{T(x) \cap X_\xi^+ \cap T^b(x)^c} |F \underbrace{1_{E_+^c}}_{\subset E_n^c}|^2 \, d\xi \lesssim \lambda^2$$

Same construction for left part of tents (X_ξ^-)

$\rightsquigarrow Q_-$

In total: $Q := Q_0 \cup Q_+ \cup Q_-$

$$\tilde{E} := \bigcup_{T \in Q} T$$

$$\mu(\tilde{E}) \leq \sum_{T \in Q} \sigma(T) \lesssim \lambda^{-2}$$

$$\text{outsup}_{\tilde{E}^c} \, S^b(f) \lesssim \lambda$$

both L_∞ & L_2 size

$$\Rightarrow \mu(S^b(f) > \lambda) \lesssim \lambda^{-2} \|f\|_2$$

Q.E.D.